SPECTRAL CHARACTERIZATION OF TWO SIDED IDEALS IN $B(H)$*

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ABSTRACT

In this paper we prove that an element $A \in B(H)$ belongs to a proper two sided ideal in $B(H)$ if and only if

$$\sigma(A + T) \cap \sigma(T) \neq \varnothing \ \forall T \in B(H)$$

In particular, if H is separable, this property completely characterizes compact operators on H.

1. **Introduction**

Let us suppose that H denotes a complex Hilbert space, $B(H)$ the ring of bounded linear operators that map H into H, and for each $C \in B(H)$ let $\sigma(C)$ denote the spectrum of C. An element $A \in B(H)$ is called non-perturbing if for each element $T \in B(H)$ it is true that

$$\sigma(A + T) \cap \sigma(T) \neq \varnothing. \tag{1.1}$$

The principal result of this paper is to prove that an element $A \in B(H)$ is non-perturbing if and only if A belongs to a proper two sided ideal of $B(H)$.

The case in which H is an infinite dimensional space is presented in Section 2 together with several lemmas. In this situation, if H is separable, then the validity of (1.1) for every $T \in B(H)$ is a necessary and sufficient condition for A to be compact (Corollary B of Section 2). In Section 3 we state our principal result in a form independent of the dimension of H and indicate some extension of our theory to von-Neumann algebras.

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2. **Infinite dimensional case**

We shall continue to use the notation established in Section 1 together with other notation that is commonly used in operator theory. We shall need the following lemmas.

LEMMA 1. *If H is an infinite dimensional Hilbert space and*

$$M = \{T;\ T \in B(H) \text{ and } \dim(\text{Range } T) < \dim H\}^*,$$

then M is a proper two sided ideal of $B(H)$.

DEFINITION 1. Suppose H is an infinite dimensional Hilbert space, $T \in B(H)$, $T = UK$ the polar decomposition of T, and $K = \int_0^\infty \lambda\, dP(\lambda)$ the spectral decomposition of K. For $\varepsilon > 0$ we define $R(T,\varepsilon) = \dim(\text{Range } P[\varepsilon, \infty))$.

LEMMA 2. *If H is an infinite dimensional Hilbert space and*

$$M_c = \{T;\ T \in B(H) \text{ and } R(T,\varepsilon) < \dim H \text{ for each } \varepsilon > 0\},$$

then M_c is the unique maximal (hence closed) two sided ideal of $B(H)$.

The preceeding Lemmas appear in various related forms in the literature (cf. [3]), so we shall not present their proofs.

LEMMA 3. *Let H be an infinite dimensional Hilbert space, $0 \neq A \in B(H)$, $A = A^*$, $A = \int_{-\infty}^{\infty} \lambda dP(\lambda)$ the spectral decomposition of A, and suppose A does not belong to a proper two sided ideal of $B(H)$. Then there exist a real number $\alpha \in \sigma(A)$ such that $0 \neq \alpha$ and if Δ is an open set containing α then* $\dim\{\text{Range } P(\Delta)\} = \dim H$.

PROOF. Suppose the Lemma is false and that for each non zero element, say α, of $\sigma(A)$ there is an open set Δ_α containing α such that $\dim\{\text{Range } P(\Delta_\alpha)\} < \dim H$. Hence, for each $\alpha \neq 0$ and $\alpha \in \sigma(A)$, $P(\Delta_\alpha) \in M$ and, consequently, $A \in M_c$ which contradicts our hypothesis.

LEMMA 4. *If A satisfies the hypothesis of Lemma 3, then there exists a self adjoint operator $T \in B(H)$ such that $\sigma(T - iA)$ contains no real number.*

PROOF. In view of Lemma 3 we may assume that there exists a real number α in $\sigma(A)$ such that $\dim\{\text{Range } P(\Delta)\} = \dim H$ for every open set Δ containing α and, moreover, without loss of generality we assume $\alpha = 1$. We shall now deal with two special cases and, in these instances, write A_0 (respectively T_0) in place of A (respectively T). In the first case suppose

* By the dimension of a subspace we always mean the topological dimension.

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{bmatrix}$$

and acts on a direct sum of an arbitrary Hilbert space with itself; in this case set

$$T_0 = \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}.$$

A direct and simple computation shows that $T_0 - iA_0 - \lambda I$ is invertible if λ is real.

In the second special case suppose A_0 acts on a sum of a relatively small Hilbert space with a relatively large Hilbert space (we ask the reader to be patient at this point inasmuch as the remainder of the proof will clear up the use of these special cases and the use of the terms "small" and "large"). We now re-express A_0 as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

so that it acts on the direct sum of three Hilbert spaces, the first two having the same dimension and the third larger. We re-express T_0 as

$$\begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Again $\sigma(T_0 - iA_0)$ contains no real number. Now using the semi-continuity of spectrum (cf. [2], pp. 53) if Δ is an open set containing $\sigma(T_0 - iA_0)$ and Δ contains no real number, then there is $\varepsilon > 0$ such that if $K \in B(H)$ and $\| K - (T_0 - iA_0) \| < \varepsilon$, then $\sigma(K) \subset \Delta$; i.e. $\sigma(K)$ contains no real number. We assume $\varepsilon < \frac{1}{4}$ and it shall remain fixed throughout this argument.

Returning to the general case we write

$$H = H' \oplus H'' \oplus H'''$$

and

$$A = A' \oplus A'' \oplus A'''$$

where $\sigma(A') \subset (-\varepsilon, \varepsilon)$, $\sigma(A'') \subset (1 - \varepsilon, 1 + \varepsilon)$, and $\sigma(A''') = \sigma(A) \setminus (\sigma(A') \cup \sigma(A''))$. Let us first get the conclusion in the case where $A''' = 0$ and $H''' = \{0\}$. Hence

$$A = \begin{bmatrix} A' & 0 \\ 0 & A'' \end{bmatrix}.$$

Consider the A_0

$$\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

of the corresponding size and set T equal to the T_0 that worked in that case. Since

$$\|(T - iA) - (T_0 - iA_0)\| = \|i(A - A_0)\| \leqq \max(\|A'\|, \|I - A''\|) < \varepsilon,$$

we have that $\sigma(T - iA)$ contains no real number. In case $A''' \neq 0$, H''' will be a reducing space for T so that we can define T to be zero on H''' and reduce the argument to the first case of this paragraph. We call the reader's attention to the fact that the first special case of the first paragraph of this proof occurs if $\dim H' + \dim H'' = \dim H$, and the second case occurs if $\dim H' < \dim H''$.

We come now to our principal theorem.

THEOREM A. *Suppose H is an infinite dimensional and complex Hilbert space and $A \in B(H)$, $A = A_1 + iA_2$ ($A_1 = (A + A^*)/2$ and $A_2 = (A - A^*)/2i$) such that A_2 belongs to no proper two sided ideal. Then there exists $T_0 \in B(H)$ such that*

$$\sigma(T_0 - A) \cap \sigma(T_0) = \varnothing$$

PROOF. A_2 satisfies the hypothesis of Lemma 4. Hence, there exists $T \in B(H)$ such that $\sigma(T - iA_2)$ contains no real number and T is selfadjoint. If we set $T_0 = T + A_1$, then T_0 is selfadjoint and $\sigma(T_0)$ contains only real numbers. Hence, $\sigma(T_0 - A) = \sigma(T - iA_2)$ is disjoint from $\sigma(T_0)$.

THEOREM B. *Suppose H satisfies the hypothesis of the preceding theorem and $A \in B(H)$. Then*

$$\sigma(T + A) \cap \sigma(T) \neq \varnothing \tag{2.8}$$

for each $T \in B(H)$ if and only if A belongs to a proper two sided ideal of $B(H)$.

PROOF. If A belongs to a proper two sided ideal in $B(H)$, then (Lemma 2) $A \in M_c$. If ϕ denotes the natural homomorphism of $B(H)/M_c = \bar{B}$, then for $T \in B(H)$, $\phi(T + A)$ and $\phi(T)$ have the same spectrum as elements of $\bar{B}$; we denote this spectrum by $\sigma(\phi(T))_B$. Since $\sigma(T))_B \subset \sigma(T)$, $\sigma(\phi(T+A))_B \subset \sigma(T+A)$ and $\sigma(\phi(T)_B = \sigma(T+A))_B$, (2.8) is satisfied.

If A does not belong to a proper two sided ideal of $B(H)$, then at least one of A_1 and A_2 ($A = A_1 + iA_2$, $A_1 = (A + A^*)/2$) does not belong to the ideal M_c (Lemma 2). Since the property expressed by (2.8) is invariant under scalar multiplication of A, we can assume $A_2 \notin M_c$. It follows now from Theorem A that (2.8) is not satisfied for every $T \in B(H)$.

This concludes the proof of Theorem B.

In the case where H is infinite dimensional and separable, then according to Calkin (cf. [1]) M_c is the set of compact operators in $B(H)$. Hence, combining the Calkin result with Theorem B leads to a "spectral characterization" of the compact operators. We state this as

COROLLARY B. *Suppose H is an infinite dimensional and separable Hilbert space and $A \in B(H)$. Then A satisfies the relation* (2.8) *for each $T \in B(H)$ if and only if A is a compact operator.*

3. The general case

In the preceding section we studied the case where H is infinite dimensional in order to exploit the ideal structure of $B(H)$. In the case where H is finite dimensional, $B(H)$ contains no proper two sided ideal; in this situation, for $0 \neq A \in B(H)$, it is always possible to construct a nilpotent operator T (i.e. $\rho(T)$ contains only zero) such that $0 \notin \sigma(A + T)$ so that (2.8) fails to hold.

We shall now give a brief description of how to construct the aforementioned operator T when the operator A is given in advance as an operator on the n-dimensional Hilbert space.

If A is a scalar multiple of the identity, put $T = 0$. In all other cases there exists a vector f such that f and Af, are linearly independent. Using a basis where f and Af are the first and last terms allows us to write A in the form

$$A = \begin{bmatrix} 0 & & & \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \\ 1 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

where a_{ni}, $i = 2, \cdots, n$ are of no importance). Now let M denote the upper triangular $(n-1) \times (n-1)$ matrix whose diagonal entry in position i is $1 - b_{ii}$ and (i,j) entry for $1 \leq i < j < n-1$ is $-b_{ij}$. If

$$T = \begin{bmatrix} 0 & & \\ 0 & & \\ \vdots & M & \\ 0 & 0 \cdots 0 & \end{bmatrix},$$

then T is nilpotent and $A + T$ is invertible.

Hence, our principal result is

THEOREM C. *If H is a complex Hilbert space and A a nonzero element of $B(H)$, then there exists a $T \in B(H)$ such that A completely perturbs T (i.e. $\sigma(T+A) \cap \sigma(T) = \varnothing$) if and only if A belongs to no proper two sided ideal of $B(H)$.*

We shall conclude this paper with some remarks relating Theorem C and a von Neumann factor R. If R is a I_∞ or II_∞ type factor, then Definition 1 is valid with the usual dimension replaced by the von Neumann dimension; in this situation Lemmas 3 and 4 are valid and the operator T constructed in Lemma 4 belongs to R. In case R is of type III_∞, R contains no proper two sided ideal and the construction of Lemma 4 can be carried on for every nonzero element of R with the corresponding T belonging to R. In case R is a finite type I_n, then $R = B(H)$ for some finite dimensional H and Theorem C applies. We have not been able to settle the case when R is a II_1 factor; in this case we conjecture that for $0 \neq A \in R$, there exists $T \in R$ such that $\sigma(T + \mathrm{A}) \cap \sigma(T) = \varnothing$.

Note added in the revision. Since the submission of our original paper several mathematicians have communicated to us different proofs. In two instances we felt the new proofs, in addition to being shorter and less computational, exhibited what is really involved. In view of this we felt obliged to recall our paper and use the two new proofs. The two in case is the new proof to Lemma 4 (due to P. R. Halmos) and the proof of the finite dimensional case (due to A. Brown and P. R. Halmos). Brown and Halmos actually prove the theorem for an $n \times n$ matrix over an arbitrary field.

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